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On the Gauge Variance of Action Functions Under Transformations on Space-Time

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Abstract

The problem of the gauge variance or invariance of action functions in classical mechanics is discussed from a group and path-theoretic viewpoint. By using the elementary theory of the cohomology of groups, criteria are introduced which enable one to decide when action functions gauge variant under a kinematical group are equivalent to action functions invariant under the transformations of the group. The criteria are applied to action functions gauge variant under Lorentz and Galilei transformations, where we deduce that any action function gauge variant under the Lorentz group is equivalent to an action function invariant under Lorentz transformations, whilst action functions gauge variant under the Galilei group are not necessarily equivalent to Galilei-invariant action functions. It is also shown that any action function gauge variant in a more restricted fashion which we define in the text, is necessarily equivalent to a 'kinetic-energy' action.

1. *Introduction*

The motivation for this article was provided by a recent paper of J. M. Lévy-Leblond. In that article Lévy-Leblond (1969) was able to link the problem of the gauge variance or invariance of Lagrangians to the quantum mechanical problem of computing the ray representations of the kinematical group of the underlying space-time. The latter problem, the calculation of the ray representation of a group, is a problem in the cohomology theory of groups. Here we discuss the gauge variance of action functions under transformations on the underlying space-time using group and path theoretic methods, particularly the cohomology of groups. We shall show how it is of more relevance to the gauge variance or invariance of action functions to compute a first cohomology group of a certain cochain complex (Lévy-Leblond calculated via a related second cohomology group). The connexion between our first cohomology group and the second cohomology group used by Lévy-Leblond has been established elsewhere (Whiston, 1969) and will not be pursued here.

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For convenience of calculation we shall define four types of gauge variance of an action function under a transformation group Q. These are respectively Q-variance, weak Q-variance, Q-covariance and weak Q-covariance, a progressively stronger sequence of 'invariances' of action functions under transformations of Q. We shall see that action functions which are weakly Q-variant or weakly Q-covariant are equivalent to action functions invariant under Q. The main purpose of this article is to establish when Q-variance or Q-covariance imply weak Q-variance or weak Q-covariance.

The first part of our discussion is concerned with the definition of an action function as a function on paths in a topological space. Here we shall discuss (in a non-rigorous fashion) paths in topological spaces and groups only as far as we need to define our problem. The second part centres on the definition of the above-mentioned types of variance of action functions under a group Q of transformations on the topological Q-module carrying a representation of Q on whose paths an action function is defined. Lastly, we apply our formalism to the study of the gauge variance of action functions in classical relativistic and non-relativistic mechanics.

(1) *Paths* (*see Spanier* Algebraic Topology)

Let T be a topological space. One says that $\tau_1, \tau_2 \in T$ are connected by a path on T if there is an

$$\alpha \in \mathrm{Map}(I, T)$$

(I is the unit closed interval in $\mathbf{R}$, and map means continuous function) such that $\alpha(0) = \tau_1$, $\alpha(1) = \tau_2$. Denote all such paths by $\Omega(\tau_1, \tau_2)$. If $\alpha \in \Omega(\tau_1, \tau_2)$ we shall write $\alpha: \tau_1 \to \tau_2$. The relation $\tau_1 \sim \tau_2$ iff $\Omega(\tau_1, \tau_2) \neq \phi$ is an equivalence relation in T and splits T into path components. T is said to be pathwise connected if there exists a point τ of T whose component is T. We shall assume from now on that T is pathwise connected. Given $\alpha: \tau_1 \to \tau_2$ we shall write $s(\alpha) = \tau_1 = \alpha(0)$ and $e(\alpha) = \tau_2 = \alpha(1)$ for the 'start' and 'end' of α respectively. The collection $\Omega(T)$ of all paths in T has a binary operation defined as follows. Given $\alpha_1, \alpha_2 \in \Omega(T)$ such that $e(\alpha_1) = s(\alpha_2)$ we may define a path $\alpha_2 \wedge \alpha_1$ from $s(\alpha_1)$ to $e(\alpha_2)$ using the glueing lemma. Without going into detail, $\alpha_2 \wedge \alpha_1$ is $\alpha_1: s(\alpha_1) \to e(\alpha_1)$ and then $\alpha_2: s(\alpha_2) = e(\alpha_1) \to e(\alpha_2)$.

Now let us suppose that T is a pathwise connected topological Q-module for some topological group Q. Then T is an Abelian group and there is a homomorphism

$$p \in \mathrm{Hom}(Q, \mathrm{Aut}(T))$$
$$p(q): \tau \mapsto q.\tau \qquad \text{for } q \in Q$$

such that $(q, \tau) \mapsto q.\tau$ is a continuous function. Because T is a Q-module, Q operates on the family of paths $\Omega(T)$ in T. Given $\alpha: \tau_1 \to \tau_2$ there is a path $q.\alpha: q.\tau_1 \to q.\tau_2$ defined by $q.\alpha: t \mapsto q.\alpha(t)$ for all $t \in I$. Clearly $q.\alpha$ is a map and hence a path. This action of Q in $\Omega(T)$ also induces an action of Q in the Abelian group Fun($\Omega(T)$, **R**) of real-valued functions on $\Omega(T)$ under pointwise addition.

Now T is a topological group and also operates on $\Omega(T)$. Given $\alpha \in \Omega(T)$ there is a path $\tau.\alpha$ from $\tau.s(\alpha)$ to $\tau.e(\alpha)$ for any $\tau \in T$ defined by

$$\tau.\alpha : t \mapsto \tau\alpha(t) \qquad \text{for } t \in I.$$

Because of this we may translate any path to the identity

$$\Omega(\tau_1, \tau_2) = \tau_1.\Omega(e_T, \tau_1^{-1}.\tau_2)$$

Given $\tau \in T$ let us write $\Lambda_T(\tau) \equiv \Omega(e_T, \tau)$ and $\Lambda(T)$ for the family of all $\Lambda_T(\tau)$. The family $\Lambda(T)$ has a binary relation related to the one we define above. Let $\alpha_1 \in \Lambda_T(\tau_1)$, $\alpha_2 \in \Lambda_T(\tau_2)$ and define $\alpha_1\alpha_2 : I \to T$ by the recipie

$$\alpha_1\alpha_2 : t \mapsto \alpha_1(t)\alpha_2(t) \qquad \text{for } t \in I$$

Note that $s(\alpha_1\alpha_2) = s(\alpha_1)s(\alpha_2) = e_T$ and $e(\alpha_1\alpha_2) = e(\alpha_1)e(\alpha_2)$. Clearly $\alpha_1\alpha_2$ is a map so that $\alpha_1\alpha_2 \in \Lambda_T(\tau_1\tau_2)$. In fact we have

$$\Lambda_T(\tau_1).\Lambda_T(\tau_2) = \Lambda_T(\tau_1\tau_2)$$

for any $\tau_1, \tau_2 \in T$. Moreover, the set of loops at e_T:—$\Lambda_T(e_T)$ is an identity for this (associative) composition and we may denote $\Lambda_T(\tau^{-1})$ by $\Lambda_T(\tau)^{-1}$. In this case Λ_T is an isomorphism

$$\Lambda_T : T \cong \Lambda(T)$$

2. '*Action Functions*'

We shall define an action function as follows.

Definition (1)

An 'action function' is any 'semigroup' function

$$A \in \text{Fun}(\Omega(T), \mathbf{R})$$

from the set of paths in a pathwise connected space T to the reals. If T is a topological group, the sets $\Omega(T)$ and $\Lambda(T)$ coincide and moreover $\Lambda(T)$ is a group isomorphic to T, so that an action function is definable also on $\Lambda(T)$. We now define the notion of equivalence of two action functions, noting that the sets Fun($\Omega(T)$, **R**), Fun($\Lambda(T)$, **R**) are Abelian groups under pointwise addition; and are Q-modules when T is a topological Q-module.

Definition (2)

Two action functions A_1 and A_2 are equivalent, $A_1 \simeq A_2$, iff $\forall\ \alpha \in \Omega(T)$,

$$A_1(\alpha) = A_2(\alpha) + \phi(\alpha)$$

where ϕ is a function such that

$$\phi(\alpha_1) = \phi(\alpha_2) \qquad \text{if } \alpha_2 \in \Omega(s(\alpha_1), e(\alpha_1))$$

ϕ depends only on the end points of any path.

The function $\phi \in \mathrm{Fun}(\Omega(T), \mathbf{R})$ hence defines a function ϕ_A^* on the collection $\Lambda(T)$ of sets of paths by

$$\phi^*(\Lambda_T(\tau)) = \phi(\alpha) \qquad \text{when } \alpha \in \Lambda_T(\tau),\ e(\alpha) = \tau$$

Recall that Q operates on the groups $\mathrm{Fun}(\Omega(T), \mathbf{R})$ and $\mathrm{Fun}(\Lambda(T), \mathbf{R})$ we shall now define the notion of Q-variance of action functions.

Definition (3)

An action function A is called Q-variant iff

$$A^q \simeq A\ \forall\ q \in Q$$

This means that $A^q(\alpha) = A(\alpha) + \phi_A(q)(\alpha)$ at each path $\alpha \in \Omega(T)$ where $\phi_A \in \mathrm{Fun}(Q, \mathrm{Fun}(\Omega(T), \mathbf{R}))$. As we saw above, ϕ_A defines a function $\phi_A^* \in \mathrm{Fun}(Q, \mathrm{Fun}(\Lambda(T), \mathbf{R}))$ where $\phi_A^*(q)(\Lambda_T(\tau)) = \phi_A(q)(\alpha), e(\alpha) = \tau$. The following proposition is a trivial consequence of the definitions and the material in the appendix:

Proposition (4)

If A is Q-variant, then

$$\phi_A \in Z_p{}^1(Q, \mathrm{Fun}(\Omega(T), \mathbf{R}))$$
$$(\phi_A^* \in Z_p{}^1(Q, \mathrm{Fun}(\Lambda(T), \mathbf{R}))).$$

Proof: By the definition of ϕ_A,

$$A^q(\alpha) = A(\alpha) + \phi_A(q)(\alpha)\ \forall\ \alpha \in \Omega(T)$$

Therefore, using the fact that $A^{q_1 q_2} = (A^{q_1})^{q_2}$, we obtain

$$A^{q_1 q_2}(\alpha) = A(\alpha) + \phi_A(q_1 q_2)(\alpha) = A(\alpha) + \phi_A(q_1)(\alpha) + \phi_A(q_2)^{q_1}(\alpha)$$

Therefore

$$\phi_A(q_1) + \phi_A(q_2)^{q_1} - \phi_A(q_1 q_2) = 0$$

or

$$\phi_A \in Z_p{}^1(Q, \mathrm{Fun}(\Omega(T), \mathbf{R}))$$

In exactly a similar way, one shows that

$$\phi_A^* \in Z_p{}^1(Q, \mathrm{Fun}(\Lambda(T), \mathbf{R}))$$

This proposition motivates the below definition:

Definition (5)

An action function A is said to be weakly Q-variant iff it is Q-variant and

$$\phi_A \in B_p{}^1(Q, \mathrm{Fun}(\Omega(T), \mathbf{R}))$$

and

$$(\phi_A^* \in B_p{}^1(Q, \mathrm{Fun}(\Lambda(T), \mathbf{R})))$$

The importance of this definition is shown by the proposition below.

Proposition (6)

If an action function is weakly Q-variant, it is equivalent to an action invariant under Q.

Proof: If an action function A is weakly Q-variant then

$$A^q(\alpha) = A(\alpha) + \phi_A(q)(\alpha) \qquad \text{where } \phi_A \in B_p{}^1(Q, \mathrm{Fun}(\Omega(T), \mathbf{R}))$$

But then $\phi_A(q) = \psi_A{}^q - \psi_A$ where $\psi \in \mathrm{Fun}(\Omega(T), \mathbf{R})$ by the definition of a one coboundary. Consequently

$$A^q(\alpha) - \psi_A{}^q(\alpha) = A(\alpha) - \psi_A(\alpha)$$

Thus the action function $A' \equiv A - \psi_A$ which is equivalent to A is invariant under the transformations of Q.

We can now establish a criterion for the coincidence of Q-variance with weak Q-variance:

Proposition (7)

Any Q-variant action function is weakly Q-variant and hence equivalent to an action function invariant under Q iff

$$H_p{}^1(Q, \mathrm{Fun}(\Lambda(T), \mathbf{R})) = 0$$

Proof: If $H_p{}^1(Q, \mathrm{Fun}(\Lambda(T), \mathbf{R})) = 0$, $B_p{}^1(Q, \mathrm{Fun}(\Lambda(T), \mathbf{R})) = Z_p{}^1(Q, \mathrm{Fun}(\Lambda(T), \mathbf{R}))$ so that if A is Q-variant $\phi_A^* \in Z_p{}^1(Q, \mathrm{Fun}(\Lambda(T), \mathbf{R})) \Rightarrow \phi_A^* \in B_p(Q, \mathrm{Fun}(\Lambda(T), \mathbf{R}))$ so that A is weakly Q-variant. Obviously weak Q-variance always implies Q-variance.

In the above proposition then, we have established a criterion for the coincidence of Q-variance with weak Q-variance. The criterion is that if one can prove that $H_p{}^1(Q, \mathrm{Fun}(\Lambda(T), \mathbf{R})) = 0$, Q-variance implies weak Q-variance. It is, however, practically impossible to calculate such a cohomology group because of the size of $\mathrm{Fun}(\Lambda(T), \mathbf{R}))$. We have to introduce some more restrictions on the gauge functions to facilitate calculation. To this end, we define a more restricted type of gauge variance under Q:– Q-covariance.

Definition (8)

An action function A is said to be Q-covariant iff it is Q-variant and

$$\phi_A^* \in Z_p{}^1(Q, \mathrm{Hom}(\Lambda(T), \mathbf{R}))$$

Definition (9)

An action function A is said to be weakly Q-covariant iff it is Q-covariant and

$$\phi_A^* \in B_p{}^1(Q, \mathrm{Hom}(\Lambda(T), \mathbf{R}))$$

These definitions immediately lead to propositions (10) and (11).

Proposition (10)

Any action function which is weakly Q-covariant is equivalent to an action function invariant under Q.

Proposition (11)

Q-covariance and weak Q-covariance coincide iff

$$H_p{}^1(Q, \mathrm{Hom}(\Lambda(T), \mathbf{R})) = 0$$

The proofs are exactly analogous to the proofs of propositions (6) and (7). For this more restrictive type of Q-variance then, we require that the gauge function ϕ_A^* of an action function which is Q-covariant satisfy

$$\phi_A^*(\Lambda_T(\tau_1 . \tau_2)) = \phi_A^*(\Lambda_T(\tau_1)) + \phi_A(\Lambda_T(\tau_2))$$

In the following, proposition (11) is used to examine the relation between Q-covariance and weak Q-covariance in classical relativistic and non-relativistic mechanics.

3. *Q-Covariance of Action Functions on Space-Time*

The criteria for the topological properties of the arenas of classical relativistic mechanics and non-relativistic mechanics used in part (1) are certainly met. For relativistic mechanics, the choice is usually $\mathbf{R}^4$ with the Euclidean topology. In this case $\mathbf{R}^4$ is a pathwise connected, simply connected topological $GL(4, \mathbf{R})$ module. Classical space-time is the topological product $\mathbf{R}^3 \times \mathbf{R}$ which is homomorphic to $\mathbf{R}^4$. We take Minkowski-space with the $\mathbf{R}^4$ topology to be a homogeneous space of the Poincaré group and consider the variance of action functions under $0(1,3;\mathbf{R}) = L$ the homogeneous Lorentz group. Classical space-time is a homogeneous space of the Galilei group $G = (\mathbf{R}^3 \otimes \mathbf{R}) \boxtimes p\ \mathrm{E}(3, \mathbf{R})v$. Here $\mathrm{E}(3, \mathbf{R})v$ is the homogeneous Galilei group of rigid motions in 'velocity space' and $p \in \mathrm{Hom}(\mathrm{E}(3, \mathbf{R})v$, $\mathrm{Aut}(\mathbf{R}^3 \otimes \mathbf{R}))$ is just

$$p(\mathbf{v}, R) : (\mathbf{x}, t) \mapsto (R\mathbf{x} + \mathbf{v}t, t)$$

where $\mathbf{v}$ is a 'velocity boost'; R a rotation, and $(\mathbf{x}, t) \in \mathbf{R}^2 \otimes \mathbf{R}$ a spatio-temporal translation. We define classical space-time as $\mathbf{R}^3 \times \mathbf{R}$.

Theorem (1)

$H_p{}^1(L, \mathrm{Hom}(\mathbf{R}^4, \mathbf{R})) = 0$, or L-covariance coincides with weak L-covariance.

Proof: We make use of the fact that the centre of L, the Lorentz group is $\mathbf{Z}(2)PT$, the two-element cyclic group generated by the space-time reflection

$$PT : x \mapsto -x \ \forall \ x \in \mathbf{R}^4$$

Now if $\phi \in Z_p{}^1(L, \mathrm{Hom}(\mathbf{R}^4, \mathbf{R}))$, the following conditions must be true:

(i) $\phi(\Lambda)(x_1 + x_2) = \phi(\Lambda)(x_1) + \phi(\Lambda)(x_2)\ \forall\ \Lambda \in L, x_1, x_2 \in \mathbf{R}^4$

(ii) $\phi(\Lambda_1 \Lambda_2)(x) = \phi(\Lambda_1)(x) + \phi(\Lambda_2)(\Lambda_1^{-1} x)\ \forall\ x \in \mathbf{R}^4, \Lambda_1 \Lambda_2 \in L.$

But $C(L) = Z(2)PT$, so that $PT\Lambda = \Lambda PT\ \forall\ \Lambda \in L$. Consequently

$$\phi(PT\Lambda)(x) = \phi(PT)(x) + \phi(\Lambda)(-x) = \phi(\Lambda)(x) + \phi(PT)(\Lambda^{-1} x)$$

This means that

$$\phi(\Lambda)(x) = \tfrac{1}{2}(\phi(PT)(x) - \phi(PT)(\Lambda^{-1} x))$$

i.e.

$$\phi(\Lambda) = -\tfrac{1}{2}(\delta(\phi PT))(\Lambda)$$

where $\phi(PT) \in \mathrm{Hom}(\mathbf{R}^4, \mathbf{R})$. Thus $\phi \in B_p{}^1(L, \mathrm{Hom}(\mathbf{R}^4, \mathbf{R}))$ and consequently

$$B_p{}^1(L, \mathrm{Hom}(\mathbf{R}^4, \mathbf{R})) = Z_p{}^1(L, \mathrm{Hom}(\mathbf{R}^4, \mathbf{R}))$$

or

$$H_p{}^1(L, \mathrm{Hom}(\mathbf{R}^4, \mathbf{R})) = 0$$

Theorem (2)

$H_p{}^1(E(3, \mathbf{R})v, \mathrm{Hom}(\mathbf{R}^3 \otimes \mathbf{R})) \neq 0$, so that for Galilei mechanics $E(3, \mathbf{R})v$ covariance and weak $E(3, \mathbf{R})v$ covariance do not coincide.

Proof: We establish $H_p{}^1(E(3, \mathbf{R})v, \mathrm{Hom}(\mathbf{R}^3 \otimes \mathbf{R}, \mathbf{R})) \neq 0$ by exhibiting a cocycle of $Z_p{}^1(E(3, \mathbf{R})v, \mathrm{Hom}(\mathbf{R}^3 \otimes \mathbf{R}, \mathbf{R}))$ which is not an element of the group $B_p{}^1(E(3, \mathbf{R})v, \mathrm{Hom}(\mathbf{R}^3 \otimes \mathbf{R}, \mathbf{R}))$.

Now any element ϕ of $Z_p{}^1(E(3, \mathbf{R})v, \mathrm{Hom}(\mathbf{R}^3 \otimes \mathbf{R}, \mathbf{R}))$ must satisfy

$$\begin{aligned}\phi(\mathbf{v}, R)((\mathbf{x}_1, t_1)(\mathbf{x}_2, t_2)) = \phi(\mathbf{v}, R)((\mathbf{x}_1 + \mathbf{x}_2, t_1 + t_2)) &= \phi(\mathbf{v}, R)(\mathbf{x}_1, t_1) \\ &\quad + \phi(\mathbf{v}, R)(\mathbf{x}_2, t_2)\end{aligned} \tag{3.1}$$

$$\begin{aligned}\phi((\mathbf{v}_1, R_1)(\mathbf{v}_2, R_2))(\mathbf{x}, t) = \phi(\mathbf{v}_1 + R_1 \mathbf{v}_2, R_1 R_2)(\mathbf{x}, t) &= \phi(\mathbf{v}_1, R_1)(\mathbf{x} + t) \\ &\quad + \phi(\mathbf{v}_2, R_2)(R_1^{-1}(\mathbf{x} - \mathbf{v}_1 t_1, t))\end{aligned} \tag{3.2}$$

Because of the algebraic structure of the Galilei group (Whiston, 1969), there exist the following four monomorphisms:

$$\begin{array}{ll} i_1\colon \mathbf{R}^3 \rightarrowtail \mathbf{R}^3 \otimes \mathbf{R} & i_1\colon \mathbf{x} \mapsto (\mathbf{x}, 0) \\ i_2\colon \mathbf{R} \rightarrowtail \mathbf{R}^3 \otimes \mathbf{R} & i_2\colon t \mapsto (0, t) \\ j_1\colon \mathbf{R}_v{}^3 \rightarrowtail E(3, \mathbf{R})v & j_1\colon \mathbf{v} \mapsto (\mathbf{v}, e) \\ j_2\colon 0(3, \mathbf{R}) \rightarrowtail E(3, \mathbf{R})v & j_2\colon R \mapsto (0, R) \end{array}$$

Via these monomorphisms, one can define cochains

$$\phi_1 \in \mathrm{Fun}(E(3, \mathbf{R})v, \mathrm{Hom}(\mathbf{R}^3, \mathbf{R}));\ \phi_1(\mathbf{v}, R) = \phi(\mathbf{v}, R) \circ i$$
$$\phi_2 \in \mathrm{Fun}(E(3, \mathbf{R})v, \mathrm{Hom}(\mathbf{R}, \mathbf{R}));\ \phi_2(\mathbf{v}, R) = \phi_2(\mathbf{v}, R) \circ i_2$$

By condition (3.1) on ϕ we can write equation (3.3)

$$\phi(\mathbf{v}, R)(\mathbf{x}, t) = \phi_1(\mathbf{v}, R)(\mathbf{x}) + \phi_2(\mathbf{v}, R)(t) \tag{3.3}$$

Further if we define cochains:

$$\phi^1 \in \mathrm{Fun}(\mathbf{R}_v{}^3, \mathrm{Hom}(\mathbf{R}^3 \otimes \mathbf{R}, \mathbf{R}));\ \phi^1 = \phi \circ j_1$$

$$\phi^2 \in \mathrm{Fun}(0(3, \mathbf{R}), \mathrm{Hom}(\mathbf{R}^3 \otimes \mathbf{R}, \mathbf{R}));\ \phi \circ j_2 = \phi^2$$

then we obtain the following results via (3.1) and (3.2) above:

$$\phi^2(R)(\mathbf{x}, t) = \phi^2(R) \circ i_1(\mathbf{x}) + \phi^2(R) \circ i_2(t) \tag{3.4}$$

$$\phi^1(\mathbf{v})(\mathbf{x}, t) = \phi^1(\mathbf{v}) \circ i_1(\mathbf{x}) + \phi^1(\mathbf{v}) \circ i_2(t) \tag{3.5}$$

$$\phi^2(R_1 R_2) \circ i_1(x) = \phi^2(R_1) \circ i_1(\mathbf{x}) + \phi^2(R_2) \circ i_1(R_1^{-1} x) \tag{3.6}$$

$$\phi^1(\mathbf{v}_1 + \mathbf{v}_2) \circ i_1(\mathbf{x}) = \phi^1(\mathbf{v}_1) \circ i_1(\mathbf{x}) + \phi^1(\mathbf{v}_2) \circ i_1(\mathbf{x}) \tag{3.7}$$

$$\phi^1(\mathbf{v}_1 + \mathbf{v}_2) \circ i_2(t) = \phi^1(\mathbf{v}_1) \circ i_2(t) + \phi^1(\mathbf{v}_2) \circ i_2(t) - \phi^1(\mathbf{v}_2) \circ i_1(\mathbf{v}_1 t) \tag{3.8}$$

Let us define a cochain $\phi_2{}^2 \in \mathrm{Fun}(0(3, \mathbf{R}), \mathrm{Hom}(\mathbf{R}, \mathbf{R}))$ by $\phi_2{}^2(R) = \phi^2(R) \circ i_2$. Then it follows from (3.2) that in fact

$$\phi_2{}^2 \in \mathrm{Hom}(0(3, \mathbf{R}), \mathrm{Hom}(\mathbf{R}, \mathbf{R}))$$

Since $0(3, \mathbf{R}) = \mathbf{Z}(2)p \otimes S0(3, \mathbf{R})$ (where $\mathbf{Z}(2)p$ is the two-element cyclic group generated by the parity operator P, and $S0(3, \mathbf{R})$ is the proper rotation group) and $\mathrm{Hom}(\mathbf{R}, \mathbf{R})$ is torsion free and Abelian, $\mathrm{Hom}(0(3, \mathbf{R}), \mathrm{Hom}(\mathbf{R}, \mathbf{R})) = 0$, so $\phi_2{}^2$ vanishes. But $\phi_1{}^2 \in \mathrm{Fun}(0(3, \mathbf{R}), \mathrm{Hom}(\mathbf{R}^3, \mathbf{R}))$ defined by $\phi_1{}^2(R) = \phi^2(R) \circ i_1$ is by (3.6), a cocycle of $Z_p{}^1(0(3, \mathbf{R}), \mathrm{Hom}(\mathbf{R}^3, \mathbf{R}))$. In a way similar to the proof of theorem (1) one can show that

$$Z_p{}^1(0(3, \mathbf{R}), \mathrm{Hom}(\mathbf{R}^3, \mathbf{R})) = B_p{}^1(0(3, \mathbf{R}), \mathrm{Hom}(\mathbf{R}^3, \mathbf{R}))$$

Consequently $\phi_1{}^2$ is a one coboundary of the latter group. It follows that it gives rise to a coboundary of $B_p{}^1(E(3, \mathbf{R})v, \mathrm{Hom}(\mathbf{R}^3 \otimes \mathbf{R}, \mathbf{R}))$ so that we may equate $\phi_1{}^2$ to zero. But $\phi^2 = \phi_1{}^2 + \phi_2{}^2$, so that $\phi^2 = 0$. Thus, so far, we have shown that

$$\phi(\mathbf{v}, R)(\mathbf{x}, t) = \phi^1(\mathbf{v})(\mathbf{x}, t) \tag{3.9}$$

where

$$\phi^1(R\mathbf{v})(R\mathbf{x}, t) = \phi^1(\mathbf{v})(\mathbf{x}, t).\ \forall\ R \in 0(3, \mathbf{R}) \tag{3.10}$$

Let us define more cochains:

$$\phi_1{}^1 \in \mathrm{Fun}(\mathbf{R}_v{}^3, \mathrm{Hom}(\mathbf{R}^3, \mathbf{R}));\ \phi_1{}^1(\mathbf{v}) = \phi^1(\mathbf{v}) \circ i_1$$

$$\phi_2{}^1 \in \mathrm{Fun}(\mathbf{R}_v{}^3, \mathrm{Hom}(\mathbf{R}, \mathbf{R}));\ \phi_2{}^1(\mathbf{v}) = \phi^1(\mathbf{v}) \circ i_2$$

Now, by (3.7), $\phi_1{}^1 \in \mathrm{Hom}(\mathbf{R}_v{}^3, \mathrm{Hom}(\mathbf{R}^3, \mathbf{R})) \simeq \mathrm{Hom}(\mathbf{R}_v{}^3 \otimes_{\mathbf{Z}} \mathbf{R}^3, \mathbf{R})$ where now $\mathbf{R}^3 \otimes_{\mathbf{Z}} \mathbf{R}$ is the tensor product of Abelian groups. By (3.10), $\phi_1{}^1$ is invariant under $0(3, \mathbf{R})$, so $\phi_1{}^1 \in \mathrm{Hom}(\mathbf{R}_v{}^3 \otimes_{\mathbf{R}} \mathbf{R}^3, \mathbf{R})$ and must be linearly related to the inner product:

$$\phi_1{}^1(\mathbf{v}_1)(\mathbf{x}) = \alpha \mathbf{v} \cdot \mathbf{x} \qquad \text{where } \alpha \in \mathbf{R} \tag{3.11}$$

The function $\phi_2{}^1$ must satisfy the following identities:

$$\phi_2{}^1(\mathbf{v}+\mathbf{v}_2)(t) = \phi_2{}^1(\mathbf{v}_1)(t) + \phi_2{}^1(\mathbf{v}_2)(t) - \phi_1{}^1(\mathbf{v}_2)(\mathbf{v}_1\, t) \qquad \text{by (3.8)} \tag{3.12}$$

$$\phi_2{}^1(\mathbf{v})(t_1+t_2) = \phi_2{}^1(\mathbf{v})(t_1) + \phi_2{}^1(\mathbf{v})(t_2) \tag{3.13}$$

$$\phi_2{}^1(R\mathbf{v})(t) = \phi_2{}^1(\mathbf{v})(t)\ \forall\ R \in 0(3,\mathbf{R}) \tag{3.14}$$

Consequently, we must have $\phi_2{}^1(\mathbf{v})(t) = (\alpha/2)\gamma(\mathbf{v}^2)\,t$, where γ is some function from $\mathbf{R}$ to $\mathbf{R}$ which, by virtue of (3.11) and (3.12) must satisfy

$$\gamma((\mathbf{v}_1+\mathbf{v}_2)^2) = \gamma(\mathbf{v}_1{}^2) + \gamma(\mathbf{v}_2{}^2) - 2\mathbf{v}_1\cdot\mathbf{v}_2 \tag{3.15}$$

Thus γ must be $\gamma(\mathbf{v}^2) = \mathbf{v}^2$ and hence we have

$$\phi_2{}^1(\mathbf{v})(t) = \frac{\alpha}{2}\mathbf{v}^2\, t$$

and

$$\phi^1(\mathbf{v})(\mathbf{x},t) = \phi(\mathbf{v},R)(\mathbf{x},t) = \frac{\alpha}{2}(\mathbf{v}^2\, t - 2\mathbf{v}\cdot\mathbf{x})$$

Thus

$$Z_p{}^1(E(3,\mathbf{R})\,v, \mathrm{Hom}(\mathbf{R}^3\otimes\mathbf{R})) \neq B_p{}^1(E(3,\mathbf{R}), \mathrm{Hom}(\mathbf{R}^3\otimes\mathbf{R},\mathbf{R}))$$

since Φ defined by $\Phi(\alpha)(\mathbf{v},R)(\mathbf{x},t) = (\alpha/2)(\mathbf{v}^2 t - 2\mathbf{v}\cdot\mathbf{x})$ is not a 1 co-boundary. In fact any one cocycle is cohomologous to such a cocycle since we choose a general element ϕ of the group of one cocycles. ∎

Corollary (3)

Any action function A which is $E(3,\mathbf{R})v$ covariant is equivalent to a 'kinetic energy' function.

Proof: Suppose that A is an action function which is $E(3,\mathbf{R})v$ covariant. Then $\forall\ (\mathbf{v},R) \in E(3,\mathbf{R})v$,

$$A^{(v,R)}(\mathbf{x},\tau) = A(R^{-1}(\mathbf{x}-\mathbf{v}\tau),\tau) = A(\mathbf{x},\tau) + \phi(\mathbf{v},R)(\mathbf{x},\tau)$$

where $(\mathbf{x},\tau)$ is a path in $\mathbf{R}^3\otimes\mathbf{R}$ and $\phi \in Z_p{}^1(E(3,\mathbf{R})v, \mathrm{Hom}(\mathbf{R}^3\otimes\mathbf{R},\mathbf{R}))$. We saw in the proof of theorem (2) that A must be equivalent to A' where

$$A'^{(v,R)}(\mathbf{x},\tau) = A'(R^{-1}(\mathbf{x}-\mathbf{v}\tau),\tau) = A'(\mathbf{x},\tau) + \frac{\alpha}{2}(\mathbf{v}^2\,\tau - 2\mathbf{v}\cdot\mathbf{x})$$

Thus, choosing $R = e$, $\mathbf{v} = \mathbf{x}/\tau$, we obtain:

$$A'(\mathbf{x},\tau) = A'(0,\tau) - \frac{\alpha}{2}(\mathbf{x})^2\,\tau^{-1} = A'(0,\tau) - \frac{\alpha}{2}\left(\frac{x}{\tau}\right)^2\tau$$

If τ is interpreted as the time, then $A'(0,\tau)$ is dependent only on the end point of the path, so that:

$$A'(\mathbf{x},\tau) = -\frac{\alpha}{2}(\dot{\mathbf{x}}^2)\,\tau, \text{ a classical kinetic energy function} \qquad ∎$$

Conclusions

Criteria have been established which enable one to decide when Q-variance of an action function implies weak Q-variance or Q-invariance, and when Q-covariance implies weak Q-covariance. Using these criteria, we have been able to show that action functions which are Q-covariant for $Q \cong L$, the Lorentz group, are equivalent to Q-invariant action functions. Action functions Q-covariant for Q-isomorphic to the homogeneous Galilei group are not necessarily weakly Q-covariant, but they are equivalent to actions which are classical kinetic-energy action functions.

Appendix

Algebraic Cohomology of Groups

Any sequence $C = \langle C^n, \delta^n \rangle n \in \mathbf{Z}^+$ of Abelian groups C^n and homomorphisms $\delta^n \in \mathrm{Hom}(C^n, C^{n+1})$ where $\delta^n \circ \delta^{n-1} = 0 \; \forall \; n > 0$, $(\delta^{-1} \equiv 0)$ is called a zero or semi-exact sequence. $\delta^n \circ \delta^{n-1} = 0$ means that $\mathrm{Im}(\delta^{n-1}) = B^n \subset Z^n = \mathrm{Ker}(\delta^n)$. B^n is called the group of n-coboundaries, Z^n the group of n-cocycles and C^n the group of n-cochains of C. The group

$$H^n = Z^n / B^n$$

is called the n-dimensional cohomology group of C. If $H^n = 0 \; \forall \; n \in Z^+$, C is called an exact sequence. Two cocycles equivalent modulus B^n are called 'cohomologous'.

Given that K is a Q-module (K *a priori* Abelian) with $p \in \mathrm{Hom}(Q, \mathrm{Aut}(K))$ defining the action of Q in K, a zero sequence $Cp(Q, K)$ can be defined as follows. We write

$$C_p{}^n(Q, K) = \mathrm{Fun}(Q^n, K)$$

which are Abelian groups under pointwise addition, and define

$$\delta^n \in \mathrm{Hom}(C_p{}^n(Q, K), C_p{}^{n+1}(Q, K)) \text{ via } \delta^{-1} = 0$$

and

$$\delta^n(f)(q_1, \ldots q_{n+1}) = p(q_1)(f(q_2, \ldots q_{n+1})) + (-1)^{n+1} f(q_1, \ldots q_n)$$
$$+ \sum_{i=1}^{n+1} (-1)^i f(q_1, \ldots q_i q_{i+1}, \ldots q_{n+1})$$

Then one can show (Maclane, 1967) that $\delta^n \circ \delta^{n-1} = 0$. The group

$$H_p{}^n(Q, K) = \mathrm{Ker}(\delta^n)/\mathrm{Im}(\delta^{n-1}) = Z_p{}^n(Q, K)/B_p{}^n(Q, K)$$

is called the n-dimensional cohomology group of Q in K. Thus, the group $H^1 p(Q, K)$ is the additive group of all 1-cocycles of Q in K:

$$(f \in Z_p{}^1(Q, K) \text{ iff } \delta(f)(q_1, q_2) = p(q_1)(f(q_2)) - f(q_1 q_2) + f(q) = 0)$$

modulus the subgroup $B_p{}^1(Q, K)$ of 1-coboundaries:

$$(f' \in B_p{}^1(Q, K) \text{ iff } \exists \; k \in K \vdash f'(q) = \delta(k)(q) = p(q)(k) - k)$$

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